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# TWO SOLUTIONS OF THE THREE-DIMENSIONAL PROBLEM OF LIMIT WAVES ON THE SURFACE OF A PONDERABLE FLUID*

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An example is given of the non-uniqueness of the solution of a stationary problem of waves in a vessel with a source or sink at the bottom. The range of the Froude numbers is determined, within which two solutions exist, with the limit waves of Stokes type appearing in one of them. A numerical method of solving non-linear wave problems is given.

The non-uniqueness of solutions of certain non-linear problems of waves on the surface of a ponderable fluid of finite depth was predicted in /1/. An example of such a problem was found at a later stage; the formation of unified waves in a flow past a protuberence on the bottom of a channel was computed in /2/, and it was found that two solutions corresponded to a single value of the Froude number Fr constructed relative to the depth of the unperturbed flow. The dependence of the flow characteristics on another parameter, namely on the velocity $v_m$ at the crest of the wave, is, however, unique.

The present paper gives an example of a non-unique solution for the limiting case of non-linearity of the wave ($v_m \equiv 0$) over a certain range of the values of Fr.

1. Axisymmetric problems are the most suitable models of the problems of the theory of flows with free boundaries, since they match the small batches of computations, approximately equal to those in the plane problem, with the faster (spatial) decay of the perturbations. In the present paper we consider the representative problem of the flow ina vessel containing an ideal ponderable incompressible fluid flowing out through an opening situated on the axis of symmetry.

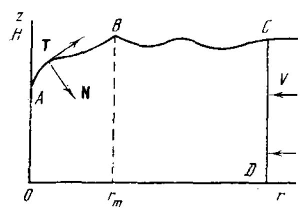


Fig.1

The flow scheme is shown in Fig.1, which shows the half-plane $z$, $r$ of the meridional cross-section of the vessel. The force of gravity is parallel to the $z$ axis, and the fluid occupies the vessel with a bottom and a side surface whose cross-sections are $OD$ and $DC$ respectively. The fluid is bounded above by a free surface whose form is not known in advance. The concentrated sink of capacity $Q^*$ is situated at the point $O$ which is the origin of coordinates, and is compensated by a ring of sources lying on $DC$, so that the resulting flow is steady.

The above hyrodynamic problem reduces to a non-linear boundary-value problem for a harmonic function, i.e. for the velocity potential $\Phi$, typical for wave theory:

$$\Delta\Phi = 0,\ (\nabla\Phi, N)\,|_S = V \tag{1.1}$$

$$(\nabla\Phi, \nabla\Phi) + 2gz\,|_{AC} = \mathrm{const} \tag{1.2}$$

Here $g$ is the acceleration due to gravity, $N$ is the normal to the boundary $S$ of the flow, and the velocity of fluid flow $V$ through this boundary is non-zero only at the point $O$ and on $DC$.

2. The scheme shown in Fig.1 corresponds to only one of the solutions of problem (1.1)-(1.2). Other solutions, in addition to such a solution with a "funnel" and a corner point at the tip of the wave are possible. In the second scheme we have the corresponding concept of a free surface at a height $H$ on the axis of symmetry and a decay $U = |\nabla\Phi|$ as $r$ increases. In the third scheme this concept is combined with a non-decrease in velocity as $r \to \infty$, but unlike the plane problem /3/, in the axisymmetric problem for this scheme $z \to 0$ as $r \to \infty$.

When considering the form of the free surface with a break, we must know the angle at the tip of the wave. Since from the Bernoulli integral it follows that

$$U = \sqrt{2g\,(H - z)} \tag{2.1}$$

we have, in the neighbourhood of the tip at $|z - H| \ll H/2,\ r \sim H$,

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$$\left|\frac{1}{r}\frac{\partial\Phi}{\partial r}\right| < \frac{U}{r} = \frac{\sqrt{2g(H-z)}}{r} \ll \frac{\sqrt{g}}{\sqrt{2(H-z)}} = \frac{\partial U}{\partial z} \sim \left|\frac{\partial^2\Phi}{\partial z^2}\right|$$

This implies that we can carry out a qualitative analysis of the flow near the tip, using the methods of plane theory, namely a conformal mapping. When mapping the angle $\alpha$ with apex at the point $\{z = H, r = r_m\}$ onto the half-plane $\zeta$, we have /4/ $z_1^{\pi} \sim \zeta^{\alpha}$, and for the complex velocity $dW/dz_1 = (dW/d\zeta)(dz_1/d\zeta)^{-1} \sim (H-z)^{\pi/\alpha-1}$. From (2.1) on the other hand, we have $dW/dz_1 \sim (H-z)^{1/2}$, $\alpha = 2\pi/3$.

In the problem in question the fluid particles near the free surface move relative to various phases of the wave which has a fixed position in space, just as in the case of the waves generated by a ship behind a stationary model of a ship in a water tank. The flow in this case depends, generally speaking, on two parameters, namely the Froude number $\mathrm{Fr} = Q^* H^{-5/2} (2g\rho)^{-1/2}$, and the ratio $\xi_c = r_c/H$ ($\rho$ is the density of the fluid and $r_c$ is the radius of the vessel), but in all computations carried out the quantity $1/\xi_c = 0.06$, was fixed, i.e. the problem solved was, in fact, a one-parameter problem.

3. Below we have used the suggestion made in /5/ that the numerical methods used successfully in /5, 6/ in the non-linear theory of cavitational flows should be modified for use with wave problems. Here, the non-linearity of the problem is also governed by the presence of an a priori unknown free boundary of the fluid. Problem (1.1) has a unique solution for any form of $AC$, but in order to satisfy (1.2) also, we must select the corresponding form of $AC$. In order to do this, we shall use the process of successive approximations analogous to Newton's method for systems of transcendental equations. Having solved problem (1.1) for some initial approximation to the free boundary $AC$, we calculate the discrepancy in (1.2). Having chosen the independent variables for describing the form of the free boundary, we seek the variable perturbations, preferably small, which make the discrepancy vanish. The perturbed values are then used to establish a new boundary, problem (1.1) is again solved for this boundary, and the process is continued until the discrepancy (1.2) becomes vanishingly small everywhere on $AC$.

The use of the process of successive approximations in solving problem (1.1), does not present any fundamental difficulties. Using the increased capacity of modern computers, we can determine the derivatives of $\Phi$ with great accuracy. In order to do it, it is best to reduce their computation to two operations, namely, to determining the density $Q$ of the potential of the sample layer $\Phi_1 = \Phi - Q^*(2\pi R^*)^{-1}$ from the integral Fredholm equation of second kind /7/ which has the following from outside the corner points of $S$:

$$Q + \frac{1}{2\pi}\iint\limits_S Q\frac{\partial}{\partial N}\frac{1}{R}\,dS = -\frac{Q^*}{2\pi}\frac{\partial}{\partial N}\frac{1}{R^*} \tag{3.1}$$

and the angle dependent coefficient accompanying the first term of (3.1) at these points /8/, and to calculating the velocity components using Coulomb's formula

$$\nabla\Phi = \frac{1}{4\pi}\nabla\left(\frac{Q^*}{R} + \iint\limits_S \frac{Q}{R}\,dS\right). \tag{3.2}$$

Here $R^*$ is the distance from the central point on $S$ to the origin of coordinates, and $R$ is the distance from the control point to an arbitrary point on $S$.

When correcting the form of $S$, we must relate its deformations $h$ directed along $N$ to the perturbations in the components of $\nabla\Phi$ defining the discrepancy in (1.2). Let $S_1$ be the initial, and $S_2$ the corrected form of $AC$. The velocity components are perturbed as a result of displacing the streamlines away from $S_1$ by a distance $h$, and rotating the normal to them (i.e. changing the unit vector $N$ by a small quantity $n$), as well as a result of the perturbation in the density $Q$ of the potential of the simple layer $\Phi_1$ by a small amount $q$ (the sources with abundances of $Q$ and $q$ are distributed over the known surface $S_1$). We can write the expansions for the components of $\nabla\Phi$ on $S_2$, retaining only terms of the order of smallness not higher than the first, and denoting by $T$ the unit vector of the tangent to $S_1$

$$\begin{aligned}(\nabla\Phi, \mathrm{N}) &= \frac{\partial\Phi}{\partial N}\{Q+q\}|_{S_1} + h\frac{\partial^2\Phi}{\partial N^2}\{Q\}|_{S_1} \\ (\nabla\Phi, \mathrm{T}) &= \frac{\partial\Phi}{\partial T}\{Q+q\}|_{S_1} + h\frac{\partial^2\Phi}{\partial N\partial T}\{Q\}|_{S_1}\end{aligned} \tag{3.3}$$

The boundary conditions for $\Phi\{Q\}$, $n = -dh/\partial T$ from (1.1) hold on $AC$. After substituting (3.3) into the boundary conditions (1.1) and (1.2) on $S_2$ and taking into account the perturbations in the Archimedean force, we obtain

$$\partial\varphi/\partial N = -\partial(hU)/\partial T \tag{3.4}$$

$$\partial\varphi/\partial T + N_z\sqrt{gh}/\sqrt{2(H-z)} + \varkappa Uh = \sqrt{2g(H-z)} - U \tag{3.5}$$

Here $\varphi = \Phi\{q\}$, $\varkappa$ is the curvature of $AC$, $N_z$ is the component of $N$ (the transfer from the Cartesian coordinates attached to the normal to the curvilinear coordinates (3.4), (3.5), can be made using well-known rules /4/). Relations (3.4), (3.5) yield, after eliminating $q$, a linear relation between the discrepancy (1.2) and $h$, and the linear operators on $S_1$ used in (3.4) and (3.5) have been found already when determining $Q$ and $U$ with help of the formulas (3.1) and (3.2). When correcting $AC$, we first determine $q$ from (3.5), and then find $h$ from the ordinary differential Eq.(3.4).

In the case of the scheme with a break, the free surface is divided into two parts, the "funnel" $AB$ and the "mirror" $BC$. When determining $q$ from (3.5), a method described in /5, 6/ is used on $AB$: the Cauchy integral of the density $q/2$, determined on $AB$ is separated from $\partial\varphi/\partial T$ and inverted /7/ in the class of functions (3.5) bounded at the ends of the arc. After this it is transformed into the integral Fredholm equation of the second kind in $q$, suitable for numerial solution. The necessary condition for a bounded solution /7/ to exist now yields the corresponding value of the parameter Fr, and for such a solution we have $q(A) = q(B) = 0$.

We must also remember that from (2.1) it follows that $h(B) = 0$ near the point $B$. Therefore, we must integrate Eq.(3.4) on $AB$ beginning from the point $B$, and the asymptotic form (2.1) is used for $U$ near this point and not the value obtained from (3.2). In the general case (3.4) yields $h(A) \neq 0$ and in order to avoid the discontinuity $S$ the arc $AC$ is displaced, before carrying out the subsequent approximation, by a distance $h(A)$ in a direction parallel to the $z$ axis, i.e. we fix in the successive approximation the depth of the funnel, and gradually refine the value of its width and Fr.

The correction of the "mirror" is carried out in each approximation after correcting $AB$. In this interval we seek $q$ from (3.5), using the inversion of the Cauchy integral mentioned above and unbounded /7/ for large $r$, and carry out the integration (3.4) using (2.1) and the condition $h(B) = 0$. The unboundedness of $q$ at the point $C(r \gg H)$ is not important in this problem, since $h(c) \simeq 0$. In the case of the other scheme the "mirror" represents the whole boundary.

In order to ensure the convergence of the computations we adopt here, as in /6/, two compatible modifications of Eqs.(3.4) and (3.5); the term proportional to $\varkappa$ in (3.5) is omitted, and the factor $\alpha_p < 1$ is introduced into the left-hand side of (3.4). The point is that it is difficult to determine $\varkappa$ numerically with an error as small as the other quantities used in (3.5). On the other hand, eliminating the term $\varkappa Uh$ from (3.5) leads to an increase in the absolute values of $q$ and this must then be compensated by means of the coefficient of relaxation $\alpha_p > 0$.

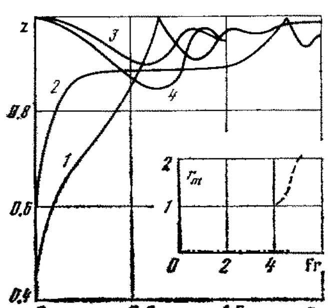


Fig.2

**4.** Fig.2 shows the results of the computations. Curves 1 and 2 corresponding to $\mathrm{Fr} = 4.07$ and $\mathrm{Fr} = 5.39$ are the meridional cross-sections of the free surface with the narrowest and the widest funnel. There are no such solutions outside the range $4.07 \leqslant \mathrm{Fr} \leqslant 5.39$, while within it all solutions are similar to those given here. Curves 3 and 4 depict the forms of $S$ without the funnels for $\mathrm{Fr} = 3.71$ and $\mathrm{Fr} = 4.23$. Such solutions exist only for $\mathrm{Fr} \leqslant 4.23$. On the "mirror" we have waves whose amplitudes decrease with distance from the axis of symmetry in all solutions,but the crests of the waves do not rise to the unperturned level $z = H$. The insert in the bottom right-hand part of Fig.2, shows the relation between the coordinate $r_m$ of the crest of the largest wave and Fr, the dashed line corresponding to the solution with a funnel and the dot-dash to the solution without it. The domain of existence of two solutions, albeit small, is much larger than the error incurred in the course of the numerical determination of Fr.

When discussing the possibilities of the physical realization of the solutions obtained, we should note, firstly, that the form of the free surface, as implied by relations (1.2), (3.1) and (3.2), does not depend on the sign of $Q^*$ (i.e. on the direction of the velocity at the point $O$). This means that the solutions of this problem are the same for a concentrated source and concentrated sink of equal strength, and secondly, since a concentrated singularity can describe the flow through a real opening of radius $R_0$ only when $R_0 \sim H/10$, it follows that for $\mathrm{Fr} \simeq 4$ and a velocity of flow $V_0$ the depths $H$ will be very small. It would obviously be more reasonable to seek an experimental confirmation not for the solutions of a one-parameter problem, but for an analogous problem with a distributed source and two parameters, namely Fr and $R_0/H$, for which there are no restrictions in the value of the ratio $R_0/H$ and such a rigid relation between $V_0$ and $H$.

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# LOWER ESTIMATES OF THE CHARACTERISTIC FREQUENCIES OF THE OSCILLATIONS OF A LIQUID WITH A FREE SURFACE IN CHANNELS OF ARBITRARY CROSS-SECTION*

V.I. TARAKANOV

Lower estimates are obtained for the leading characeristic frequency of oscillations of a liquid in a channel of arbitrary cross-section with several sections of the free surface of the liquid. The case of oscillations in the plane of the cross-section of the channel is considered. The domain occupied by the cross-section can be multiply connected and bounded by a piecewise smooth curve. The derivation of the estimates is not connected with the need to find standard domains and is not based on variational methods /1-3/.

1. The boundary eigenvalue problem

$$\Delta U \equiv U_{,xx} + U_{,yy} = 0, \quad x, y \in D \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \omega^2 U, \quad x, y \in \Gamma_\alpha; \quad \frac{\partial U}{\partial n} = 0, \quad x, y \in \Gamma/\Gamma_\alpha \tag{1.2}$$

is considered for a multiply connected domain $D \subset R^2(x, y)$ bounded by a piecewise-smooth curve $\Gamma$ consisting of a number of closed curves. $\Gamma$ has $m$ segments $\Gamma_j$ for $j = 1, 2 \cdots m$, where boundary conditions corresponding to the conditions on the free surface of the liquid are given.

$$\Gamma_j: y = h_j, \quad a_j < x < b_j, \quad j = 1, 2 \cdots m, \quad \Gamma_\alpha = \Gamma_1 \cup \Gamma_2 \cup \cdots \Gamma_m$$

It is assumed that the segments of the free surface can be placed at different levels $y = h_j$ (for example, to maintain different pressures of gases over different sections of the surface). In the general case, the segments $\Gamma_j$ may belong to different closed curves of the contour $\Gamma$. Apart from a dimensional factor, $\omega$ is identical with the characteristic frequency of oscillations of the liquid. Thus, in what follows it is called simply the frequency of characteristic oscillations of the liquid.

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